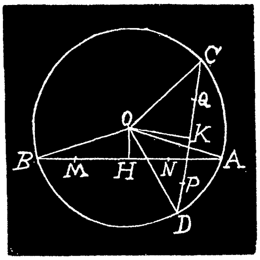
57. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

A chord is drawn through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that the quadrilateral formed by joining the extremities of the two chords will contain the center of the circle.

I. Solution by the PROPOSER.

Let M, N be the first two random points; AB the chord through them; P, Q the second two random points; C, D the chord through them; and O the center of the circle. Draw OH, OK perpendicular to AB, CD.



Let $AO=r$, $AM=w$, $MN=x$, $CP=y$, $PQ=z$, $\angle AOH=\theta$, $\angle COK=\varphi$, $\angle AOC=\psi$, and $\mu=$the angle AB makes with some fixed line.

Then $AH=r\sin\theta$, $CK=r\sin\varphi$; an element of the circle at M is $r\sin\theta d\theta dw$; at N, $d\mu x dx$; at P, $r\sin\varphi d\varphi dy$; at Q, $d\psi z dz$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and doubled; of ψ, $\pi-2\theta$ and π, and doubled; of μ, 0 and 2π; of w, 0 and $2r\sin\theta=s$; of x, 0 and w, and doubled; of y, 0 and $2r\sin\varphi=v$; and of z, 0 and y, and doubled. Hence, since the whole number of ways the four points can be taken is $\pi^4 r^8$, the required chance is

$$p=\frac{16}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\int_0^{y} r\sin\theta d\theta r\sin\varphi d\varphi d\psi d\mu dw x dx dy z dz$$

$$=\frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\sin\theta\sin\varphi d\theta d\varphi d\psi d\mu dw x dx y^2 dy$$

$$=\frac{64}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu dw x dx$$

$$=\frac{32}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu w^2 dw$$

$$=\frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi d\mu$$

$$=\frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi=\frac{1024}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\theta\sin^4\theta\sin^4\varphi d\theta d\varphi$$

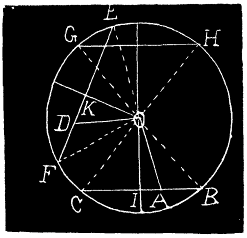
$$=\frac{128}{9\pi^3}\int_0^{\frac{1}{2}\pi}(3\theta^2-3\theta\sin\theta\cos\theta-2\theta\sin^3\theta\cos\theta)\sin^4\theta d\theta=\tfrac{2}{3}+\frac{139}{72\pi^2}.$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let A and D represent the positions of the first and third points, and sup-

pose the second and fourth to be taken somewhere upon the lines BC and EF, respectively.

Put $OC=a$, $OA=x$, $OD=y$, $\angle OAC=\theta$, $\angle IOC=\phi$, $\angle ODK=\psi$, and $\angle KOF=\omega$.



Then $a\cos\phi=x\sin\theta$, and $a\cos\omega=y\sin\psi$.

If either E or F is upon the arc HG, the quadrilateral formed by joining the extremities of the chords will contain the center of the circle.

Since the arc HG=the arc CB, the probability of this is $(\phi+\omega)/\pi$.

If $x>0$ and $<a$ the chance that the first point is taken between x and $x+dx$ is $2\pi xdx/a^2\pi=2xdx/a^2$.

If $y>0$ and $<a$ the chance that the third point is taken between the distances y and $y+dy$ is $2ydy/a^2$.

If $\theta>0$ and $<\frac{1}{2}\pi$ the chance that the second point is taken between the line BC and a second line making at A the angle $d\theta$ is $\frac{1}{2}(AC^2+AB^2)d\theta/\frac{1}{2}a^2\pi$ $=2[a^2+x^2(1-2\sin^2\theta)]d\theta/a^2\pi=2(1-2\cos^2\phi+\cos^2\phi\text{cosec}^2\theta)d\theta/\pi$.

If $\psi>0$ and $<\frac{1}{2}\pi$ the chance that the fourth point is taken between the line EF and a second line making at D the angle $d\psi$ is $2(1-2\cos^2\omega+\cos^2\omega\text{cosec}^2\psi)d\psi/\pi$.

If we suppose θ constant while x and ϕ vary, $xdx=-a^2\sin\phi\cos\phi\text{cosec}^2\theta d\phi$. When $x=0$, $\phi=\frac{1}{2}\pi$, and when $x=a$, $\phi=\frac{1}{2}\pi-\theta$. Hence the limits of integration for ϕ are $\frac{1}{2}\pi-\theta$ and $\frac{1}{2}\pi$. If we integrate first with respect to θ, the limits for θ are $\frac{1}{2}\pi-\phi$ and $\frac{1}{2}\pi$, while for ϕ they are 0 and $\frac{1}{2}\pi$.

In like manner we may substitute $-a^2\sin\omega\cos\omega\text{cosec}^2\psi d\omega$ for ydy and integrate between limits $\frac{1}{2}\pi-\omega$ and $\frac{1}{2}\pi$ for ψ, and between 0 and $\frac{1}{2}\pi$ for ω.

Hence the required probability is

$$P=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\omega}^{\frac{1}{2}\pi}(\phi+\omega)(1-2\cos^2\phi+\cos^2\phi\text{cosec}^2\theta)\times$$

$$(1-2\cos^2\omega+\cos^2\omega\text{cosec}^2\psi)\sin\phi\cos\phi\text{cosec}^2\theta\sin\omega\cos\omega\text{cosec}^2\psi d\phi d\omega d\theta d\psi$$

$$=\frac{256}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\phi+\omega)\sin^4\phi\sin^4\omega d\phi d\omega$$

$$=\frac{4}{9\pi^3}\int_0^{\frac{1}{2}\pi}(12\pi\phi+3\pi^2+16)\sin^4\phi d\phi=\tfrac{1}{2}+(8/3\pi^2)=.77019.$$

MISCELLANEOUS.

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55. Proposed by J. M. COLAW, A. M., Monterey, Va.

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number*. "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra*.]